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Extended Poincaré parasuperalgebra with central charges and invariant wave equations

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Abstract. We describe irreducible representations of the extended Poincaré parasuperalgebra (PPSA) which includes an arbitrary number N of parasupercharges, n ($n \leqslant \{N/2\}$) central charges and an internal symmetry group. We also discuss wave equations which are invariant with respect to the PPSA and propose a parasupersymmetric generalization of the Wess–Zumino model for arbitrary p and N.

1. Introduction

There are two approaches in modern physics which, in some sense, treat bosons and fermions on an equal level. One of them is called *supersymmetry* [1, 2] and presupposes using equivalence transformations which mix fermionic and bosonic states. The other approach is connected with parastatistics and paraquantization [3, 4]. In these theories a fundamental role is played by a so-called paraquantization order p, and for the limiting case $p \to 0$ parabosons are in some sense transformed to fermions and vice versa, parafermions are transformed to bosons [5, 6] (for other properties of parastatistics see, for example, [7]; a more modern treatment was presented in [8]).

Both approaches are unified in the theory called parasupersymmetric quantum mechanics (PSSQM) which appeared in 1988 [9]. It awaken undoubted interest and stimulated a number of investigations (see [10] and references cited therein). Parasupersymmetry was used in relativistic quantum mechanics [11]. Parasuperpotentials admitting Lie and non-Lie symmetries were investigated in [12] and the hidden $SU(3)$ symmetry of equations of PSSQM was established in [13].

An interesting problem of relativization of PSSQM was stated and partially solved in [14], where the grounds of parasupersymmetric quantum field theory (PPSQFT) were formulated. We note that, in fact, such a theory was discussed earlier, and the concept of the Poincaré parasupergroup (which is the symmetry group of parasupersymmetric quantum field theory) was suggested by Jarvis as long ago as 1978 [15]. For modern trends in application of parasupersymmetry in quantum field theory refer to [16].

In [14, 15] irreducible representations (IRs) of the simplest $N = 1$ Poincaré parasuperalgebra were considered and some representations corresponding to timelike and lightlike 4-momenta were discussed. All non-equivalent Hermitian IRs for timelike, lightlike and spacelike 4-momenta were described in [17].

Representations of the extended Poincaré parasuperalgebra $p(1,3;N)$ (i.e. the Poincaré parasuperalgebra with an arbitrary number N of parasupercharges, which includes the external symmetry algebra) were described in [18, 19]. Moreover, the relations between representations of $p(1,3;N)$ and representations of the pseudo-orthogonal algebras $so(p,q)$ were established and widely exploited in [19].

However, in [14–19] an important possibility was ignored which had been successfully used in *supersymmetric* quantum field theory (refer, for example, to [20, 21]). This is to make an additional extension of the PPSA by the introduction of so-called central charges which form a commutative centre of the algebra. And it is this possibility which is discussed here.

The present paper continues and in some sense completes the series of works [17–19] in which the group-theoretical foundations of PSSQFT were created. We describe IRs of the extended Poincaré parasuperalgebra with an arbitrary number N of parasupercharges, internal symmetry algebra and n central charges ($n \leqslant N/2$ for N even and $n \leqslant (N-1)/2$ for N odd).

The PPSA with the central charges is a direct generalization of the corresponding Poincaré superalgebra [22] and can be used, for example, to construct parasupersymmetric dynamical models describing parasupermultiplets with variable masses. However, this generalization is by no means trivial. An interesting new feature of the extended PPSA (in comparison with the Poincaré superalgebra) is the existence of such IRs which correspond to values of central charges larger than the doubled mass.

The other specific feature of the present paper is that we formulate the results in terms of IRs of pseudo-orthogonal groups and in terms of parasuperfields and para-Grassmanian variables as well. Finally, we present linear and nonlinear models which are invariant with respect to the PPSA.

2. Extended Poincaré parasuperalgebra

The Poincaré parasuperalgebra [14, 15, 17–19] is generated by ten generators P_μ, $J_{\mu\nu}$ ($\mu,\nu = 0,1,2,3$) of the Poincaré group, satisfying the commutation relations

$$\begin{aligned} &[P_\mu, P_\nu] = 0 \qquad [P_\mu, J_{\nu\sigma}] = \mathrm{i}(g_{\mu\nu}P_s - g_{\mu\sigma}P_\nu) \\ &[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}) \end{aligned} \tag{2.1}$$

and N parasupercharges Q_A^j, $\left(Q_A^j\right)^\dagger$ ($A = 1,2$; $j = 1,2,\ldots,N$), which satisfy the following double-commutation relations:

$$\begin{aligned} &\left[Q_A^i, \left[Q_B^j, Q_C^k\right]\right] = 2\varepsilon_{AB}Z^{ij}Q_C^k - 2\varepsilon_{AC}Z^{ik}Q_B^j \\ &\left[\left(Q_A^i\right)^\dagger, \left[\left(Q_B^j\right)^\dagger, \left(Q_C^k\right)^\dagger\right]\right] = 2\varepsilon_{AB}Z^{*ij}\left(Q_C^k\right)^\dagger - 2\varepsilon_{AC}Z^{*ik}\left(Q_B^j\right)^\dagger \\ &\left[Q_A^i, \left[Q_B^j, \left(Q_C^k\right)^\dagger\right]\right] = 2\varepsilon_{AB}Z^{ij}\left(Q_C^k\right)^\dagger - 4Q_B^j(\sigma_\mu)_{AC}P^\mu\delta^{ik} \\ &\left[\left(Q_A^i\right)^\dagger, \left[Q_B^j, \left(Q_C^k\right)^\dagger\right]\right] = 4\left(Q_C^k\right)^\dagger(\sigma_\mu)_{AB}P^\mu\delta^{ij} - 2\varepsilon_{AC}Z^{*ik}Q_B^j \end{aligned} \tag{2.2}$$

where σ_ν are the Pauli matrices, ε_{AB} is the universal spinor $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$; $(\cdot)_{AC}$ relate to matrix elements. Relations (2.1) and (2.2) include operators Z^{ij} which we call central charges.

As in the case of Poincaré superalgebra the central charges are supposed to satisfy the relations $(Z^{ij})^* = Z^{ij}$ and $Z^{ij} = -Z^{ji}$ and to commute with generators of the PPSA.

The parasupercharges are supposed to be Weyl spinors and so they should satisfy the following commutation relations with generators of the Poincaré group:

$$\begin{aligned} &\left[J_{\mu\nu}, Q_A^j\right] = -\frac{1}{2\mathrm{i}}\left(\sigma_{\mu\nu}\right)_{AB} Q_B^j, [P_\mu, Q_A^j] = 0 \\ &\left[J_{\mu\nu}, \left(Q_A^j\right)^\dagger\right] = -\frac{1}{2\mathrm{i}}\left(\sigma_{\mu\nu}^*\right)_{AB}\left(Q_B^j\right)^\dagger, [P_\mu, \left(Q_A^j\right)^\dagger] = 0 \end{aligned} \tag{2.3}$$

where $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\sigma_\mu, \sigma_\nu]$.

We stress that the extended PPSA is a direct (and natural) generalization of the Poincaré superalgebra (PSA). Indeed, the PSA also includes $10+4N$ elements satisfying (2.1) and (2.3), but instead of (2.2) supercharges Q_A^j, $\left(Q_A^j\right)^\dagger$ satisfy the following *anticommutation* relations:

$$\begin{aligned} &\left[Q_A^i, Q_B^j\right]_+ = Q_A^i Q_B^j + Q_B^j Q_A^i = \varepsilon_{AB} Z^{ij} \\ &\left[Q_A^i, \left(Q_B^j\right)^\dagger\right]_+ = 2\delta^{ij}(\sigma_\mu)_{AB} P^\mu. \end{aligned} \tag{2.4}$$

Relations (2.2) are a mere consequence of (2.4) and the converse is not true. This statement follows from the formal identity which is valid for three arbitrary operators A, B, C:

$$[A, [B, C]] = [[A, B]_+, C]_+ - [[A, C]_+, B]_+. \tag{2.5}$$

Thus, the PSA is a particular case of the more general algebraic structure called PPSA, in the same way that the usual Fermi statistics is a particular case of the parastatistics [6].

As for the Poincaré superalgebra, the PPSA can be extended by adding the generators Σ_l of the internal symmetry group which satisfy the following relations:

$$\begin{aligned} &[Q_A^i, \Sigma_l] = T_l^{ij} Q_A^j \qquad \left[\Sigma_l, \left(Q_A^i\right)^\dagger\right] = T_l^{*ij}\left(Q_A^j\right)^\dagger \\ &[\Sigma_l, \Sigma_m] = f_{lm}^k \Sigma_k \end{aligned} \tag{2.6}$$

where f_{lm}^k are structure constants of the internal symmetry group. The constants T_{lJ}^I are specified in the following.

3. Wigner little parasuperalgebra

The extended Poincaré parasuperalgebra (2.1)–(2.3), (2.6) has two main Casimir operators [19],

$$C_1 = P_\mu P^\mu \qquad C_2 = P_\mu P^\mu B_\nu B^\nu - (B_\mu P^\mu)^2 \tag{3.1}$$

where

$$B_\mu = W_\mu + X_\mu \qquad W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \qquad X_\mu = (\sigma_\mu)_{AB}(Q_A^i)^\dagger Q_B^i.$$

Here W_μ is the familiar Pauli–Lubanski vector. We will use eigenvalues of C_1, C_2 to classify IRs.

As in the case of the ordinary Poincaré group [23], IRs of the PPSA are qualitatively different for the following cases:

I. $P_\mu P^\mu = M^2 > 0$;
II. $P_\mu P^\mu = 0$;
III. $P_\mu P^\mu < 0$.

For cases I and II there exists the additional Casimir operator $C_3 = P_0/|P_0|$ whose eigenvalues are $\varepsilon = \pm 1$. Here we consider only such representations which correspond to $C_1 > 0$ and $C_3 > 0$. This class of representations will be denoted as I^+.

As follows from (2.1)–(2.3), the 4-vector B_μ satisfies the relations

$$[B_\mu, P_\nu] = 0 \qquad [B_\mu, J_{\nu\sigma}] = \mathrm{i}(g_{\mu\nu}B_\sigma - g_{\mu\sigma}B_\nu) \tag{3.2}$$

$$[B_\mu, Q_A] = \tfrac{1}{2}P_\mu Q_A \qquad [B_\mu, \bar{Q}_A] = -\tfrac{1}{2}P_\mu \bar{Q}_A \qquad [B_\mu, B_\nu] = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}P^\rho B^\sigma. \tag{3.3}$$

Considering these relations in the rest frame $P = (M, 0, 0, 0)$ we conclude that the related 3-vector

$$j_k = -\frac{B_k}{M} = S_k - \frac{X_k}{M} \qquad k = 1, 2, 3 \tag{3.4}$$

commutes with Q^j_A, $(Q^j_A)^\dagger$ and satisfies the commutation relation which characterizes the algebra $so(3)$:

$$\left[j_a, \hat{Q}^j_A\right] = \left[j_a, \hat{\hat{Q}}{}^j_A\right] = 0 \qquad [j_a, j_b] = \mathrm{i}\varepsilon_{abc}j_c. \tag{3.5}$$

If the central charges are trivial (i.e. $Z^{ij} \equiv 0$) then relations (2.2) are reduced in the rest frame to the following form:

$$\begin{aligned}
&[Q^i_A, [(Q^j_B)^\dagger, Q^k_C]] = 4M\delta_{AB}\delta^{ij}Q^k_C \\
&[(Q^i_A)^\dagger, [Q^J_B, (Q^k_C)^\dagger]] = 4M\delta_{AB}\delta^{ij}(Q^k_C)^\dagger.
\end{aligned} \tag{3.6}$$

The remaining double commutators are either equal to zero or can be reduced to (3.6).

Let the central charges be non-trivial, then using the unitary transformation

$$\begin{aligned}
&Q^i_A \longrightarrow \tilde{Q}^j_A = U^{ij}Q^j_A \qquad Z^{ij} \longrightarrow \tilde{Z}^{ij} = U^{ik}U^{jl}Z^{kl} \\
&J_{\mu\nu} \longrightarrow J_{\mu\nu} \qquad P_\mu \longrightarrow P_\mu
\end{aligned} \tag{3.7}$$

we can reduce the antisymmetric matrix Z^{ij} to the quasidiagonal representation such that all non-zero elements have the following form:

$$\tilde{Z}^{2m-1,2m} = -\tilde{Z}^{2m,2m-1} = Z_m \tag{3.8}$$

where Z_m, $m = 1, 2, \ldots, \{N/2\}$, are real and non-negative values.

Denoting $(\hat{Q}^i_A)^\dagger = \hat{\hat{Q}}{}^i_A$ and choosing a new basis

$$\begin{aligned}
&\tilde{Q}^{2m-1}_1 = \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_1 - \hat{\hat{Q}}{}^{2m}_2) \qquad \tilde{Q}^{2m-1}_2 = \frac{1}{\sqrt{2}}(\hat{Q}^{2m}_1 + \hat{\hat{Q}}{}^{2m-1}_2) \\
&\tilde{Q}^{2m}_1 = \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_1 + \hat{\hat{Q}}{}^{2m}_2) \qquad \tilde{Q}^{2m}_2 = \frac{1}{\sqrt{2}}(\hat{Q}^{2m}_1 - \hat{\hat{Q}}{}^{2m-1}_2)
\end{aligned} \tag{3.9}$$

we reduce relations (2.2) in the rest frame $P = (M, 0, 0, 0)$ to the form

$$\begin{aligned}
&[\hat{Q}^{2k-1}_A, [\hat{\hat{Q}}{}^{2m-1}_B, \hat{Q}^j_C]] = 2\delta_{AB}\delta_{km}(2M - Z_m)\hat{Q}^j_C \\
&[\hat{\hat{Q}}{}^{2k-1}_A, [\hat{Q}^{2m-1}_B, \hat{\hat{Q}}{}^j_C]] = 2\delta_{AB}\delta_{km}(2M - Z_m)\hat{\hat{Q}}{}^j_C
\end{aligned} \tag{3.10a}$$

$$\begin{aligned}
&[\hat{Q}^{2k}_A, [\hat{\hat{Q}}{}^{2m}_B, \hat{Q}^j_C]] = 2\delta_{AB}\delta_{km}(2M + Z_m)\hat{Q}^j_C \\
&[\hat{\hat{Q}}{}^{2k}_A, [\hat{Q}^{2m}_B, \hat{\hat{Q}}{}^j_C]] = 2\delta_{AB}\delta_{km}(2M + Z_m)\hat{Q}^j_C
\end{aligned} \tag{3.10b}$$

the remaining double commutators of the parasupercharges being equal to zero.

Thus the description of IRs of the extended PPSA, belonging to class I^+, reduces to a description of representations of the little Wigner parasuperalgebra (LWPSA) defined by relations (3.5), (3.6) (or (3.5), (3.10)). In accordance with (3.5) the LWPSA is a direct sum of the algebra $so(3)$ (realized by j_1, j_2 and j_3) and the algebra formed by parasupercharges.

4. Classification of IRs and the explicit form of basis elements

4.1. Various types of central charges

It is well known that Hermitian IRs of the extended Poincaré superalgebra can be defined only in the case when values of supercharges do not exceed $2M$ [22]. We will see that in the case of PPSA such IRs exist for any real values of Z_m.

We specify the following cases:

1. The central charges are trivial, i.e. $Z_m = 0, 1, 2, \ldots, \{N/2\}$.
2. The central charges are non-trivial and are smaller then $2M$.
3. The central charges are equal to $2M$.
4. The central charges are non-trivial and their values exceed $2M$.
5. The central charges are of mixed type, i.e.

$$\begin{aligned}
&0 < Z_i < 2M && i = 1, 2, \ldots, m_1 \\
&Z_{m_1+j} > 2M && j = 1, 2, \ldots, m_2 \\
&Z_{m_1+m_2+k} = 2M && k = 1, 2, \ldots, m_3 \\
&Z_{m_1+m_2+m_3+l} = 0 && l = 1, 2, \ldots, \{N/2\} - m_1 - m_2 - m_3.
\end{aligned} \tag{4.1}$$

Consider all these possibilities. Of course, cases 1–4 are particular versions of case 5.

4.2. IRs with trivial central charges

These representations are described in [17–19]. The related WLPSA can be put in the form of a direct sum of the orthogonal algebras

$$\text{WLPSA} \subset so(4N + 1) \oplus so(3) \tag{4.2}$$

and so IRs of the PPSA are induced by IRs of $so(4N + 1)$ and $so(3)$. They are labelled by the sets of numbers $(M, j, n_1, n_2, \ldots, n_{2N})$ where $n_1 \geqslant n_2 \geqslant n_3 \geqslant \cdots \geqslant n_{2N} \geqslant 0$ are both integers or half-integers, j is an integer or half-integer. The explicit form of the corresponding basis elements P_μ, $J_{\mu\nu}$, Q^i_A, $\bar{Q}^j_A$ in the momentum representation up to unitary equivalence can be chosen as [17–19]

$$\begin{aligned}
&P_0 = \varepsilon E \qquad P_a = p_a \\
&J_{ab} = x_a p_b - x_b p_a + \varepsilon_{abc} S_c \\
&J_{0a} = x_0 p_a - \frac{\mathrm{i}\varepsilon}{2}\left[\frac{\partial}{\partial p_a}, E\right]_+ - \frac{\varepsilon_{abc} p_b S_c}{E + M}.
\end{aligned} \tag{4.3a}$$

$$\begin{aligned}
&Q^i_1 = \frac{1}{\sqrt{2M(E + M)}}[\tilde{Q}^j_1(E + M + p_3) + \tilde{Q}^j_2(p_1 - \mathrm{i}p_2)] \\
&Q^i_2 = \frac{1}{\sqrt{2M(E + M)}}[\tilde{Q}^j_1(p_1 + \mathrm{i}p_2) + \tilde{Q}^j_2(E + M - p_3)] \\
&\bar{Q}^i_A = (Q^i_A)^\dagger \qquad i = 1, 2, \ldots, N
\end{aligned} \tag{4.3b}$$

where $E = \sqrt{M^2 + p^2}$, $x_a = \mathrm{i}\frac{\partial}{\partial p_a}$; S_a and $\tilde{Q}^j_A$ $(a = 1, 2, 3;\ j = 1, 2, \ldots, N;\ A = 1, 2)$ are matrices given by the following relations:

$$\begin{aligned}
&\tilde{Q}^j_1 = \sqrt{2M}(S_{\Lambda,4j-3+p} - \mathrm{i}S_{\Lambda,\,4j-2+p}) \\
&\tilde{Q}^j_2 = \sqrt{2M}(S_{\Lambda,4j-1+p} - \mathrm{i}S_{\Lambda,4j+p})
\end{aligned} \tag{4.4}$$

$$
\begin{aligned}
S_a &= S_a^{(1)} \oplus j_a \\
S_1^{(1)} &= \tfrac{1}{2}\sum_{i=1}^{s}(S_{4j+p,4j-3+p} + S_{4j-2+p,4j-1+p}) \\
S_2^{(1)} &= \tfrac{1}{2}\sum_{i=1}^{s}(S_{4j+p,4j-2+p} + S_{4j-1+p,4j-3+p}) \\
S_3^{(1)} &= \tfrac{1}{2}\sum_{i=1}^{s}(S_{4j+p,4j-1+p} + S_{4j-3+p,4j-2+p}).
\end{aligned}
\tag{4.5}
$$

Here $p = 0$ (we introduce this parameter for further references), $\Lambda = 4N+1$, $s = N$, S_{mn} are generators of the algebra $so(4N + 1)$ satisfying the following relations:

$$
[S_{kl}, S_{mn}] = -\mathrm{i}(g_{km}S_{ln} + g_{ln}S_{km} - g_{kn}S_{lm} - g_{lm}S_{km}) \tag{4.6}
$$

(where $g_{kl} = -\delta_{kl}$ and δ_{kl} is the Kronecker symbol) and realizing IR $D(n_1, n_2, \ldots, n_{2N})$, j_a are basis elements of the algebra $so(3)$, belonging to IR $D(j)$.

We see IRs of the PPSA belonging to class I^+ can be described in a rather straightforward way. For other classes of representations refer to [17–19].

4.3. IRs with $0 < Z_m < 2M$ and $Z_m > 2M$

Let all central charges be non-trivial and satisfy the condition $Z_m < 2M$. Then, using the analogy of (3.6) with (3.10), we find the general solutions of relations (3.10) in the form

$$
\begin{aligned}
\hat{Q}_1^{2m-1} &= \sqrt{2M - Z_m}(S_{\Lambda,8m-7} - \mathrm{i}S_{\Lambda,\,8m-6}) \\
\hat{Q}_2^{2m-1} &= -\sqrt{2M - Z_m}(S_{\Lambda,\,8m-5} - \mathrm{i}S_{\Lambda,8m-4}) \\
\hat{Q}_1^{2m} &= \sqrt{2M + Z_m}(S_{\Lambda,8m-3} - \mathrm{i}S_{\Lambda,\,8m-2}) \\
\hat{Q}_2^{2m} &= -\sqrt{2M + Z_m}(S_{\Lambda,8m-1} - \mathrm{i}S_{\Lambda,\,8m})
\end{aligned}
\tag{4.7}
$$

where S_{ij} are generators of $so(4N + 1)$ satisfying relations (4.6), $\Lambda = 4N + 1$, $m = 1, 2, \ldots, N/2$. Substituting (4.7) into (3.9) we obtain parasupercharges in the rest frame

$$
\begin{aligned}
\tilde{Q}_1^{2m-1} &= \sqrt{\left|M - \tfrac{1}{2}Z_m\right|}(S_{\Lambda,8m-7} + \mathrm{i}S_{\Lambda,8m-6}) + \sqrt{M + \tfrac{1}{2}Z_m}(S_{\Lambda,8m-1} + \mathrm{i}S_{\Lambda,8m}) \\
\tilde{Q}_2^{2m-1} &= \sqrt{\left|M - \tfrac{1}{2}Z_m\right|}(S_{\Lambda,8m-3} - \mathrm{i}S_{\Lambda,8m-2}) - \sqrt{M + \tfrac{1}{2}Z_m}(S_{\Lambda,8m-5} + \mathrm{i}S_{\Lambda,\,8m-4}) \\
\tilde{Q}_1^{2m} &= -\sqrt{\left|M - \tfrac{1}{2}Z_m\right|}(S_{\Lambda,8m-7} - \mathrm{i}S_{\Lambda,8m-6}) - \sqrt{M + \tfrac{1}{2}Z_m}(S_{\Lambda,8m-1} + \mathrm{i}S_{\Lambda,8m}) \\
\tilde{Q}_2^{2m} &= -\sqrt{\left|M - \tfrac{1}{2}Z_m\right|}(S_{\Lambda,8m-3} + \mathrm{i}S_{\Lambda,8m-2}) + \sqrt{M + \tfrac{1}{2}Z_m}(S_{\Lambda,8m-5} + \mathrm{i}S_{\Lambda,\,8m-4}).
\end{aligned}
\tag{4.8}
$$

If N is odd then we again have $N - 1$ parasupercharges (4.8) and the additional parasupercharge whose components are

$$
\begin{aligned}
\tilde{Q}_1^{N} &= \sqrt{2M}(S_{\Lambda,4N-3} - \mathrm{i}S_{\Lambda,\,4N-2}) \\
\tilde{Q}_2^{N} &= -\sqrt{2M}(S_{\Lambda,4N-1} - \mathrm{i}S_{\Lambda,4N}).
\end{aligned}
\tag{4.9}
$$

The related vector of spin S_a has the form

$$
S_a = (S_a^{(2)} + \tilde{S}_a) \oplus j_a \tag{4.10}
$$

where j_a are generators of the IRs $D(j)$ of algebra $so(3)$, commuting with $S_{\mu\nu}$,

$$
\begin{aligned}
S_1^{(2)} &= \tfrac{1}{2}\sum_{i=1}^{t}(S_{8i-7,8i-4} + S_{8i-6,8i-5} + S_{8i,8i-3} + S_{8i-1,8i-2}) \\
S_2^{(2)} &= \tfrac{1}{2}\sum_{i=1}^{t}(S_{8i-6,8i-4} + S_{8i-5,8i-7} + S_{8i,8i-2} + S_{8i-3,8i-1}) \\
S_3^{(2)} &= \tfrac{1}{2}\sum_{i=1}^{t}(S_{8i-7,8i-6} + S_{8i-5,8i-4} + S_{8i-3,8i-2} + S_{8i-1,8i}) \\
t &= 2\{N/2\}.
\end{aligned}
\qquad (4.11)
$$

For N even $\tilde{S}_a = 0$ while for the case of odd N

$$
\begin{aligned}
\tilde{S}_1 &= \tfrac{1}{2}(S_{8N-7,8N-2} + S_{8N-5,8N} + S_{8N-3,8N-6} + S_{8N-1,8N-4}) \\
\tilde{S}_2 &= \tfrac{1}{2}(S_{8N-7,8N-3} + S_{8N-5,8N-1} + S_{8N-2,8N-6} + S_{8N,8N-4}) \\
\tilde{S}_3 &= \tfrac{1}{2}(S_{8N-7,8N-6} + S_{8N-5,8N-4} + S_{8N-2,8N-3} + S_{8N,8N-1}).
\end{aligned}
\qquad (4.12)
$$

In accordance with the above, the IRs of class I^+ of the extended Poincaré parasuperalgebra with central charges $Z_m < 2M$ are induced by IRs of algebra (4.2) and so are labelled by the sets of numbers $(M, j, n_1, n_2, \ldots, n_{2N}, Z_1, Z_2, \ldots, Z_{\{N/2\}})$ satisfying the relations $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{2N}$, $Z_m < 2M$ (all $n_1, n_2, \ldots$ are either integer or half-integer). The corresponding basis elements P_μ, $J_{\mu\nu}$ are given by relations (4.3) (where S_a, $a = 1, 2, 3$, have the form (4.10)) and the corresponding parasupercharges (which can be obtained starting with (4.7) by means of the Lorentz transformation) have form (4.3), where $\tilde{Q}_A^j$ are matrices given by relations (4.8) and (4.9).

In the case $Z_m > 2M$ it is convenient to search for solutions of relations (3.10) in the form

$$
\begin{aligned}
\hat{Q}_1^{2j-1} &= \sqrt{Z_j - 2M}(S_{\Lambda,8j-7} - \mathrm{i}S_{\Lambda,\,8j-6}) \\
\hat{Q}_2^{2j-1} &= -\sqrt{Z_j - 2M}(S_{\Lambda,8j-5} - \mathrm{i}S_{\Lambda,\,8j-4})
\end{aligned}
\qquad (4.13)
$$

and use the old representation (4.7) for parasupercharges with even numbers. As a result we again come to the representations (4.8)–(4.10) for parasupercharges and the spin vector, respectively. Using the technique developed in [17–19] it is possible to show that the corresponding LWPSA is induced by the following algebras:

$$
\begin{aligned}
A_{2k} &= so(2N, 2N+1) \oplus so(3) \qquad && N = 2k \\
A_{2k+1} &= so(2N-4, 2N+5) \oplus so(3) \qquad && N = 2k+1.
\end{aligned}
\qquad (4.14)
$$

Thus the related matrices $S_{\mu\nu}$ should belong to the pseudo-orthogonal algebra $so(2N, 2N+1)$ if N is even and to the algebra $so(2N-4, 2N+5)$ if N is odd. These algebras are characterized by commutation relations (4.6) where

$$
g_{\mu\nu} = \begin{cases} 0 & \mu \neq \nu \\ -1 & \mu = \nu = 8m-4, 8m-5, 8m-6, 8m-7 \\ +1 & \mu = \nu = 8m, 8m-1, 8m-2, 8m-3. \end{cases} \qquad (4.15)
$$

Thus in contrast to the PSA, there exist Hermitian IRs of the PPSA corresponding to larger then $2M$ central charges. The related basis elements of the PPSA have the form (4.3), where $S_{\mu\nu}$ are matrices belonging to an IR of the pseudo-orthogonal algebra $so(2N, 2N+1)$ if N is even and the algebra $so(2N-4, 2N+5)$ if N is odd.

4.4. The case $Z_m = 2M$

In the case $Z_m = 2M$ the expressions on the left-hand side of (3.10a) are reduced to zero. As a result for N even we obtain instead of (4.7) and (4.8)

$$\hat{Q}_A^{2m-1} = 0$$
$$\hat{Q}_A^{2m} = (-1)^{A-1}\sqrt{2M}(S_{\Lambda,4m-5+2A} - \mathrm{i}S_{\Lambda,4m-4+2A})$$

and

$$\begin{aligned}
\tilde{Q}_1^{2m-1} &= \sqrt{M}(S_{\Lambda,4m-1} + \mathrm{i}S_{\Lambda,\,4m})\\
\tilde{Q}_2^{2m-1} &= \sqrt{M}(S_{\Lambda,4m-3} - \mathrm{i}S_{\Lambda,\,4m-2})\\
\tilde{Q}_1^{2m} &= -\sqrt{M}(S_{\Lambda,4m-1} + \mathrm{i}S_{\Lambda,\,4m})\\
\tilde{Q}_1^{2m} &= \sqrt{M}(S_{\Lambda,4m-3} - \mathrm{i}S_{\Lambda,\,4m-2}).
\end{aligned} \tag{4.16}$$

Here $\Lambda = 2N + 1$, $m = 1, 2, \ldots, N/2$, $S_{\mu\nu}$ $(\mu, \nu = 1, 2, \ldots 2N + 1)$ are matrices satisfying (4.6) and belonging to IR $D(n_1, n_2, \ldots, n_N)$ of the algebra $so(2N + 1)$.

If N is odd then we again have $N - 1$ parasupercharges (4.16) (where $\Lambda = 2N + 3$) and the additional parasupercharge $\tilde{Q}_A^N$, where

$$\begin{aligned}
\tilde{Q}_1^N &= \sqrt{2M}(S_{\Lambda,2N-1} - \mathrm{i}S_{\Lambda,2N})\\
\tilde{Q}_2^N &= -\sqrt{2M}(S_{\Lambda,2N+1} - \mathrm{i}S_{\Lambda,2N+2}).
\end{aligned} \tag{4.17}$$

The corresponding spin operators have the form

$$S_a = (S_a^{(3)} + \tilde{S}_a) \oplus j_a \tag{4.18}$$

where

$$\begin{aligned}
S_1^{(3)} &= \tfrac{1}{2}\sum_{k=1}^{f}(S_{4k+p,4k-3+p} + S_{4k-2+p,4k-1+p})\\
S_2^{(3)} &= \tfrac{1}{2}\sum_{k=1}^{f}(S_{4k+p,4k-2+p} + S_{4k-1+p,4k-3+p})\\
S_3^{(3)} &= \tfrac{1}{2}\sum_{k=1}^{f}(S_{4k+p,4k-1+p} + S_{4k-3+p,4k-2+p})
\end{aligned} \tag{4.19}$$

$f = \{N/2\}$, $p = 0$, $\tilde{S}_a = 0$ for N even and $\tilde{S}_a$ are given by formulae (4.12) for N odd.

The corresponding basis elements of the PPSA in an arbitrary frame of reference are given by relations (4.3) where $\tilde{Q}_A^j$ and S_a are defined in (4.16)–(4.19).

4.5. Central charges of mixed type

This case is the most complicated one. However, for any particular interval of values of Z_m enumerated in (4.1) we can construct the basis elements of the PPSA in complete analogy with the treatment given in subsections 4.2–4.4. As a result we find $2(m_1 + m_2)$ parasupercharges of the form (4.8), $2m_3$ parasupercharges of the form (4.16) and $2(N - m_1 - m_2 - m_3)$ parasupercharges of the form (4.4), where

$$\Lambda = 4N + 1 - 2m_3. \tag{4.20}$$

The related symbols S_{kl} used in (4.3), (4.4), (4.8) and (4.16) denote basis elements of the Lie algebra of the pseudo-orthogonal group $SO(4N - 2m_3 - 2m_2 + 1, 2m_2)$.

The corresponding vector of spin has the form

$$S_a = (S_a^{(1)} + S_a^{(2)} + S_a^{(3)}) \oplus j_a$$

where $S_a^{(1)}$, $S_a^{(2)}$ and $S_a^{(3)}$ are given by relations (4.5), (4.11) and (4.19), respectively, where $p = 8(m_1 + m_2)$, $s = N - 2m_1 - 2m_2$, $t = m_1 + m_2$, $f = m_3$. Finally, the related basis elements of the PPSA in an arbitrary frame of reference are again given by relations (4.3), where $\tilde{Q}_A^j$ and S_a are matrices defined in the present section.

Thus in the case of central charges of mixed type the LWPSA can be enclosed into the Lie algebra

$$A = so(2m_2, 4N - 2m_3 - 2m_2 + 1) \oplus so(3) \tag{4.21}$$

which induces IRs of the PPSA.

5. Internal symmetries

5.1. General analysis

Let us demonstrate that the space of IR of the extended PPSA is a carrier space for the internal symmetry group, and construct explicitly the related group generators.

If central charges are trivial then commutation relations (2.1)–(2.3) are transparently invariant with respect to unitary transformations

$$\begin{aligned} &Q_A^j \longrightarrow U^{jk} Q_A^k \qquad J_{\mu\nu} \longrightarrow J_{\mu\nu} \qquad P_\mu \longrightarrow P_\mu \\ &U^{jk}(U^{ik})^\dagger = \delta^{ij} \end{aligned} \tag{5.1}$$

and so the corresponding PPSA admits the internal symmetry group $U(N)$.

If central charges are non-trivial then the internal symmetry group is less extended. Indeed, consider the first of relations (2.2) for $A = C = 1$, $B = 2$:

$$[Q_1^i, [Q_2^j, Q_1^k]] = 2Z^{ij} Q_1^k. \tag{5.2}$$

Calculating commutators of the left- and right-hand sides of (5.2) with Σ_l and using (2.6) we arrive at the following condition:

$$T_l^{ij} Z^{jk} = T_l^{kj} Z^{ji} = (T_l^{ij} Z^{jk})^\dagger. \tag{5.3}$$

In other words, a product of a generator of the internal symmetry group with the matrix of central charges should be a symmetric and Hermitian matrix.

In the case of N even and all $Z_m \neq 0$ relation (5.3) specifies the algebra $sp(\{N/2\})$. For N odd or for Z_m of combined type (4.1) the matrix Z^{kl} is equivalent to the direct sum of the invertible antisymmetric matrix and the zero matrix and the related conditions (5.1) and (5.3) specify the direct sum of algebras $sp(m_1 + m_2 + m_3) \oplus u(N - 2(m_1 + m_2 + m_3))$.

Thus the structure of internal symmetries for the PPSA is clear and is analogous to the case of PSA [22]. We shall find generators of the internal symmetry algebra in an explicit form and show that this algebra is a subalgebra of (4.21). We will consider successively all the cases enumerated in subsection 4.1.

5.2. Internal symmetries for trivial central charges

If $Z^{kl} \equiv 0$ then IRs of the PPSA are induced by the algebra (4.2). To find the internal symmetry algebra it is sufficient to define the maximal subalgebra of (4.2), whose generators Σ^{ab} satisfy the condition

$$[\Sigma^{ab}, S_c] = 0 \tag{5.4}$$

where S_c is the operator of spin (4.5). Indeed, condition (5.4) selects matrices which commute with generators P_μ, $J_{\mu\nu}$ (4.3) of the Poincaré group. A complete set of such matrices can be chosen in the following form:

$$\begin{aligned}\Sigma^{ab} = {} & \tfrac{1}{2}(S_{4a-2+p,4b-3+p} + S_{4a+p,4b-1+p} - S_{4a-3+p,4b-2+p} - S_{4a-1+p,4b+p}) \\ & -\tfrac{1}{2}\mathrm{i}(S_{4a-3+p,4b-3+p} + S_{4a-2+p,4b-2+p} + S_{4a-1+p,4b-1+p} + S_{4a+p,4b+p})\end{aligned} \tag{5.5}$$

where $a, b = 1, 2, \ldots, N$, $p = 0$, $S_{\mu\nu} = -S_{\nu\mu}$, $S_{\mu\nu} \subset so(4N+1)$.

Operators (5.5) satisfy (5.4) and fulfil the following relations:

$$[\Sigma^{ab}, \tilde{Q}^c_A] = \delta^{ac}\tilde{Q}^c_A \tag{5.6}$$

$$[\Sigma^{ab}, \Sigma^{cd}] = \delta^{ad}\Sigma^{bc} - \delta^{bc}\Sigma^{ad} \tag{5.7}$$

and so form a basis of the internal symmetry algebra $u(N)$.

5.3. Internal symmetries for $Z_m \neq 2M$

If $0 < Z_m < 2M$ and N is even then IRs again are induced by the algebra (4.2). The internal symmetry algebra is a subalgebra of (4.2) whose basis elements are A^{kn}, $B^{kn} = B^{nk}$ and $C^{kn} = C^{nk}$, where

$$\begin{aligned}
A^{kn} &= \frac{f_n^-}{2f_k^-}a(k,n) + \frac{f_k^+}{2f_n^+}a(k+\tfrac{1}{2}, n+\tfrac{1}{2}) + \frac{M(Z_k - Z_n)}{2f_k^- f_n^+ Z_n Z_k}a(k, n+\tfrac{1}{2}) \\
B^{kn} &= U^{kn} + U^{nk} \\
U^{kn} &= \frac{f_n^-}{2f_k^-}b(k,n) + \frac{1}{2}\frac{f_k^+}{f_n^+}b(k+\tfrac{1}{2}, n+\tfrac{1}{2}) + \frac{M(Z_k - Z_n)}{2f_k^- f_n^+ Z_k Z_n}b(k, n+\tfrac{1}{2}) \\
C^{kn} &= V^{kn} + V^{nk} \\
V^{kn} &= \frac{1}{2}\frac{f_n^-}{f_k^-}b^\dagger(k,n) + \frac{f_k^+}{f_n^+}b^\dagger(k+\tfrac{1}{2}, n+\tfrac{1}{2}) + \frac{M(Z_k - Z_n)}{2f_k^- f_n^+ Z_k Z_n}b^\dagger(k+\tfrac{1}{2}, n) \\
f_m^\pm &= \sqrt{\left|\frac{M}{Z_m} \pm \frac{1}{2}\right|} \\
a(k,n) &= S_{8n-6,8k-7} + S_{8k-6,8n-7} + S_{8n-5,8k-4} + S_{8k-5,8n-4} \\
&\quad +\mathrm{i}(S_{8n-5,8k-5} + S_{8n-4,8k-4} + S_{8n-7,8k-7} + S_{8n-6,8k-6}) \\
b(k,n) &= S_{8n-4,8k-7} + S_{8k-6,8n-5} + \mathrm{i}(S_{8n-5,8k-7} + S_{8n-4,8k-6}).
\end{aligned} \tag{5.8}$$

Here $k, n = 1, 2, \ldots, \{N/2\}$, $S_{\mu\nu} \subset so(4N+1)$. Formulae (5.8) define $\frac{1}{2}N(N-1)$ linearly independent matrices A^{kn}, B^{kn} and C^{kn} which satisfy the following commutation relations:

$$
\begin{aligned}
&[A^{kn}, \tilde{Q}_A^{2j-1}] = \delta_{kj}\sqrt{\frac{Z_k}{Z_n}}\tilde{Q}_A^{2n-1} \\
&[A^{kn}, \tilde{Q}_A^{2j}] = -\delta_{nj}\sqrt{\frac{Z_n}{Z_k}}\tilde{Q}_A^{2k} \qquad [A^{kn}, S_a] = 0 \\
&[B^{kn}, \tilde{Q}_A^{2j-1}] = \delta_{kj}\sqrt{\frac{Z_k}{Z_n}}\tilde{Q}_A^{2n} + \delta_{kn}\sqrt{\frac{Z_n}{Z_k}}\tilde{Q}_A^{2k} \\
&[B^{kn}, \tilde{Q}_A^{2j}] = 0 \qquad [B^{kn}, S_a] = 0 \\
&[C^{kn}, \tilde{Q}^{2j}] = \delta_{kj}\sqrt{\frac{Z_k}{Z_n}}\tilde{Q}_A^{2n-1} + \delta_{kn}\sqrt{\frac{Z_n}{Z_k}}\tilde{Q}_A^{2k-1} \\
&[C^{kn}, \tilde{Q}_A^{2j-1}] = 0 \qquad [C^{kn}, S_a] = 0
\end{aligned}
\tag{5.9}
$$

and

$$
\begin{aligned}
&[A^{kl}, B^{mn}] = \delta^{lm}B^{kn} + \delta^{ln}B^{km} \\
&[A^{kl}, C^{mn}] = -\delta^{km}C^{ln} - \delta^{kn}C^{lm} \\
&[B^{mn}, C^{kl}] = \delta^{nk}A^{lm} + \delta^{km}A^{ln} + \delta^{lm}A^{kn} + \delta^{ln}A^{km} \\
&[B^{kl}, B^{nm}] = [C^{kl}, C^{mn}] = 0.
\end{aligned}
\tag{5.10}
$$

Relations (5.10) specify the algebra $sp(N/2)$. In accordance with (5.9) this is the internal symmetry algebra for representations of class I^+ provided N is even and $0 < Z_m < 2M$.

For the case of N odd the internal symmetry algebra is $sp((N-1)/2) \oplus u(1)$. The basis elements of $sp((N-1)/2)$ are again given by relations (5.8), where $k, n = 1, 2, \ldots (N-1)/2$ and the generator of $u(1)$ is

$$
\Lambda = S_{4N-3\,4N-2} + S_{4N-1,4N}. \tag{5.11}
$$

Formulae (5.8) and (5.11) also present basis elements of the internal symmetry algebra for the case $Z_m > 2M$. In accordance with (4.14) the related matrices $S_{\mu\nu}$ belong to $so(2N, 2N+1)$ if N is even and to $so(2N-4, 2N+5)$ if N is odd.

5.4. Internal symmetries for $Z_m = 2M$

If all $Z_m = 2M$ then in accordance with subsection 4.4 IRs of the PPSA are induced by the algebra A_N, moreover

$$
\begin{aligned}
&A_{2n} = so(2N+1) \oplus so(3) \qquad N = 2k \\
&A_{2k+1} = so(2N+3) \oplus so(3) \qquad N = 2k+1.
\end{aligned}
\tag{5.12}
$$

The related internal symmetry algebra is a subalgebra of (5.12) whose basis elements can be chosen in the form

$$
\begin{aligned}
A^{kn} = &-\tfrac{1}{2}(S_{4k-3,4n-2} + S_{4n-3,4k-2} + S_{4k,4n-1} + S_{4n,4k-1}) \\
&+\tfrac{1}{2}\mathrm{i}(S_{4k-3,4n-3} + S_{4k-2,4n-2} + S_{4k-1,4n-1} + S_{4k,4n}) \\
B^{kn} = &\ \tfrac{1}{2}(S_{4k-3,4n} + S_{4n-3,4k} + S_{4k-1,4n-2} + S_{4n-1,4k-2}) \\
&+\tfrac{1}{2}\mathrm{i}(S_{4k-3,4n-1} + S_{4k-2,4n} + S_{4n-3,4k-1} + S_{4n-2,4k}) \\
C^{kn} = &\ (B^{kn})^{\dagger}.
\end{aligned}
\tag{5.13}
$$

Here $k, n = 1, 2, \ldots, \{N/2\}$, $S_{\mu\nu} \subset so(2N+1)$ for N even and $S_{\mu\nu} \subset so(2N+3)$ if N is odd. Matrices (5.13) satisfy commutation relation (5.10) and so form a basis of algebra $sp(\{N/2\})$. They also satisfy conditions (5.9) with operators (4.16), (4.18) and so form the internal symmetry algebra. For the case of odd N the internal symmetry algebra is $sp((N-1)/2) \oplus u(1)$ and includes an additional generator (5.11).

5.5. Internal symmetries in the general case

Let central charges be of mixed type described by formula (4.1). Then IRs of the PPSA are induced by the algebra (4.21). The corresponding internal symmetry algebra (ISA) is

$$\mathrm{ISA} = sp(m_1+m_2+m_3) \oplus u(N-2m_1-2m_2-2m_3). \tag{5.14}$$

The basis elements of $u(N-2m_1-2m_2-2m_3)$ can be chosen in the form (5.5) where $p = 8(m_1+m_2+m_3)$ and $S_{\mu\nu}$ are matrices belonging to $so(2m_2, 4N-2m_2-2m_3+1)$. The basis elements of $sp(m_1+m_2+m_3)$ can be divided into three sets. The first set includes $(m_1+m_2)(2m_1+2m_2+1)$ elements defined by relations (5.8) where $k, n = 1, 2, \ldots, m_1+m_2$. The second set includes $m_3(2m_3+1)$ elements defined by relations (5.13) where $k, n = 1, 2, \ldots, m_3$. The last (third) set includes $4m_3(m_1+m_2)$ elements given by the following formulae:

$$\begin{aligned}
A^{k\,p+j} &= -\tfrac{1}{2} f_k^+ a(k,j) + \tfrac{1}{2} f_k^- a(k-\tfrac{1}{2}, j)\\
A^{p+j\,k} &= (A^{k\,p+j})^\dagger\\
B^{k\,p+j} &= B^{p+j\,k} = \tfrac{1}{2} f_k^+ b(k,j) - \tfrac{1}{2} f_k^- b(k-\tfrac{1}{2}, j)\\
C^{k\,p+j} &= C^{p+j\,k} = \tfrac{1}{2} f_k^+ b^\dagger(k,j) - \tfrac{1}{2} f_k^- b(k, j-1)
\end{aligned} \tag{5.15}$$

where $k = 1, 2, \ldots, m_1+m_2$, $j = 1, 2, \ldots, m_3$, $p = m_1+m_2$, f_k^+ and f_k^- are coefficients defined in (5.8) and

$$\begin{aligned}
a(k,j) = {} & S_{8k-3,8p+4\mathrm{i}-2} + S_{8p+4\mathrm{i}-3,2k-2} - S_{8k-1,8p+4\mathrm{i}} - S_{8p+4\mathrm{i}-1,8k}\\
& -\mathrm{i}(S_{8k-3,8p+4\mathrm{i}-3} + S_{8k-2,8p+4\mathrm{i}-2} + S_{8k-1,8p+4\mathrm{i}-1} + S_{8k,8p+4\mathrm{i}})\\
b(k,j) = {} & S_{8k-3,8p+4\mathrm{i}} - S_{8k-2,8p+4\mathrm{i}-1} + S_{8p+4\mathrm{i}-3,8k} - S_{8p+4\mathrm{i}-2,8k-1}\\
& +\mathrm{i}(S_{8k-3,8p+4\mathrm{i}-1} + S_{8k-2,8p+4\mathrm{i}} + S_{8p+4\mathrm{i}-3,8k-1} + S_{8p+4\mathrm{i}-2,8k}).
\end{aligned}$$

Thus we present explicitly basis elements of the ISA for all possible types of central charges.

6. Para-Grassmanian variables

6.1. The case $N = 1$

The description of IRs of the PPSA presented here and in [17–19] seems to be rather convenient, in as much as it has been done explicitly in terms of basis elements of IRs of familiar pseudo-orthogonal Lie algebras $so(p,q)$. However, bearing in mind possible applications to generalize (para)supersymmetric quantum field theories, we consider here a formulation of such IRs in terms of para-Grassmanian variables.

We start with the simplest case $N = 1$, when there exists only (four-component) parasupercharge, and central charges are obviously absent. Then in the rest frame parasupercharges Q_A can be expressed via parafermionic creation operators a_A^+ and annihilation operators a_A:

$$Q_A = \sqrt{2M}\, a_A \qquad (Q_A)^\dagger = \sqrt{2M}\, (a_A)^+ \tag{6.1}$$

Operators a_A and a_A^+ satisfy the double-commutation relations†

$$[[a_A, a_B], a_C] = 0 \qquad [[a_C^+, a_B], a_A] = -2\delta_{AC} a_B. \tag{6.2}$$

There exist an infinite (but countable) number of non-equivalent fields satisfying (6.2), which can be enumerated by a positive integer number p (called the order of paraquantization). For each p the operators a_A can be defined using the Green ansatz [5]

$$a_A = \sum_{\alpha=1}^{p} b_A^{(\alpha)} \tag{6.3}$$

where $b_A^{(\alpha)}$ are usual fermionic annihilation operators satisfying the following relations:

$$\begin{aligned} &[b_A^{(\alpha)}, b_B^{+(\alpha)}]_+ = \delta_{AB} && [b_A^{(\alpha)}, b_B^{(\alpha)}]_+ = 0 \\ &[b_A^{(\alpha)}, b_B^{+(\beta)}] = [b_A^{(\alpha)}, b_B^{(\beta)}] = 0 && \alpha \neq \beta. \end{aligned} \tag{6.4}$$

In addition, it is supposed that there exists a non-degenerated vacuum state $|0\rangle$ which is annihilated by any operator $b_A^{(\alpha)}$:

$$b_A^{(\alpha)}|0\rangle = 0 \qquad \text{for all } A \text{ and } \alpha. \tag{6.5}$$

Operators $b_A^{+(\alpha)}$ are defined in the domain which is dense in the Hilbert space $\Re^{(p)}$ spanned on the set of vectors $P(b^+)|0\rangle$, where P are polynomials in creation operators $b_A^{+(\alpha)}$.

Using (6.3) we can realize the representation of operators a_A and $a_A^\dagger$ in the space $\Re^{(p)}$. In addition to (6.2) we impose the following conditions:

$$a_A|0\rangle = 0 \tag{6.6}$$

$$a_A\, a_B^+|0\rangle = p\delta_{AB}|0\rangle. \tag{6.7}$$

It was shown by Greenberg and Messiah [6] that all irreducible representations of relations (6.2) in the Hilbert space with one cyclic vacuum which satisfy (6.6), must satisfy (6.7) with some positive p and are defined (up to unitary equivalence) by the conditions (6.6) and (6.7). Each such representation can be obtained using the Green ansatz (6.3).

Operators $b_A^{(\alpha)}$ and $b_A^{\dagger(\alpha)}$ can also be realized in terms of Grassmanian variables,

$$b_A^{(\alpha)} = \frac{\partial}{\partial\theta_A^{(\alpha)}} \qquad b_A^{+(\alpha)} = \theta_A^{(\alpha)} \tag{6.8}$$

which satisfy

$$\left(\theta_A^{(\alpha)}\right)^2 = 0 \qquad \frac{\partial}{\partial\theta_A^{(\alpha)}}\theta_B^{(\sigma)} + \theta_B^{(\sigma)}\frac{\partial}{\partial\theta_A^{(\alpha)}} = \delta^{\alpha\sigma}\delta_{AB}.$$

Then para-Grassmanian variables θ_A are defined via the Green ansatz (6.3):

$$a_A = \frac{\partial}{\partial\theta_A} = \sum_{\alpha=1}^{p}\frac{\partial}{\partial\theta_A^{(\alpha)}} \qquad a_A^+ = \theta_A = \sum_{\alpha=1}^{p}\theta_A^{(\alpha)}. \tag{6.9}$$

† It is worth noting that the relations (6.2) can be written in the form $[a_A, a_A^\dagger]_+ = p + 2pN_A - 2N_A^2$; $[a_A, a_A^\dagger] = p - 2N_A$, where N_A is the operator of the number of parafermions, and we can use the Dirac contour representation [24] to describe these relations.

Using representation (6.1) and (6.9) we can express basis elements (4.3) of IR of the PPSA via parafermionic creation and annihilation operators, or alternatively via para-Grassmatian variables. Indeed, in accordance with (6.3), (6.9) and (4.4) for $N = 1$

$$S_{51} = \frac{1}{2}\left(a_1 + a_1^+\right) = \frac{1}{2}\left(\frac{\partial}{\partial\theta_1} + \theta_1\right) \qquad S_{52} = \frac{1}{2\mathrm{i}}\left(a_1 - a_1^+\right) = \frac{1}{2\mathrm{i}}\left(\frac{\partial}{\partial\theta_1} - \theta_1\right)$$
$$S_{53} = \frac{1}{2}\left(a_2 + a_2^+\right) = \frac{1}{2}\left(\frac{\partial}{\partial\theta_2} + \theta_2\right) \qquad S_{54} = \frac{1}{2\mathrm{i}}\left(a_2 - a_2^+\right) = \frac{1}{2\mathrm{i}}\left(\frac{\partial}{\partial\theta_2} - \theta_2\right). \tag{6.10}$$

All the other basis elements of $so(5)$ used in (4.3) can be found using (4.6):

$$S_{kl} = -\mathrm{i}\,[S_{5k}, S_{5l}]. \tag{6.11}$$

Formulae (4.3), (6.10), (6.11) present a realization of IRs of the PPSA (characterizing by $N = 1$ and arbitrary p) in terms of para-Grassmanian variables.

Consider in detail the simplest example with a non-trivial parasupersymmetric context, i.e. $N = 1$, $p = 2$. The related fermionic creation and annihilation operators can be realized in terms of the Dirac matrices $\gamma_\mu^{(\alpha)}$

$$b_1^\alpha = \tfrac{1}{2}(\mathrm{i}\gamma_1^{(\alpha)} - \gamma_2^{(\alpha)}) \qquad b_2^\alpha = \tfrac{1}{2}(\gamma_0^{(\alpha)} - \gamma_3^{(\alpha)}) \qquad \alpha = 1, 2 \tag{6.12}$$

where $\gamma_\mu^{(1)}$ and $\gamma_\mu^{(2)}$ are commuting matrices satisfying the Clifford–Dirac algebra

$$\left[\gamma_\mu^{(1)}, \gamma_\nu^{(2)}\right] = 0 \qquad \gamma_\mu^{(\alpha)}\gamma_\nu^{(\alpha)} + \gamma_\nu^{(\alpha)}\gamma_\mu^{(\alpha)} = 2g^{\mu\nu}. \tag{6.13}$$

Supposing that these matrices are irreducible and using Shur's lemma we conclude that they are of dimension 16×16.

The corresponding relations (6.10) reduce to the form

$$S_{54} = \tfrac{1}{2}\left(\gamma_0^{(1)} + \gamma_0^{(2)}\right) = \beta_0 \qquad S_{5a} = \tfrac{1}{2}\mathrm{i}\left(\gamma_a^{(1)} + \gamma_a^{(2)}\right) = \mathrm{i}\beta_a \qquad a = 1, 2, 3. \tag{6.14}$$

The other basis elements of the algebra $so(5)$ are

$$S_{ab} = \mathrm{i}\,[\beta_a, \beta_b] \qquad S_{4a} = -\,[\beta_0, \beta_a] \qquad a, b = 1, 2, 3. \tag{6.15}$$

In view of (6.13) it is clear that matrices (6.14) and (6.15) satisfy relations (4.6). Moreover, they also satisfy the Duffin–Kemmer algebra

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\lambda\nu}\beta_\mu + g_{\mu\nu}\beta_\lambda. \tag{6.16}$$

Representation (6.14) of algebra (6.16) is reducible and equivalent to the direct sum of representations realized by 10×10, 5×5 and 1×1 matrices. The condition of the existence of a non-degenerated vacuum state for the related operators (6.3) and (6.12) reduces this representation to a 10×10 one, as will be shown in the following.

In accordance with (6.3), (6.7), (6.12), the related parafermionic annihilation and creation operators can be represented as

$$a_1 = \frac{\partial}{\partial\theta_1} = \mathrm{i}\beta_1 - \beta_2 \qquad a_2 = \frac{\partial}{\partial\theta_2} = \beta_0 - \beta_3$$
$$a_1^+ = \theta_1 = \mathrm{i}\beta_1 + \beta_2 \qquad a_2^+ = \theta_2 = \beta_0 + \beta_3. \tag{6.17}$$

Supposing that there exists a non-degenerate vacuum state $|0\rangle$, which is annihilated by a_1 and a_2, we can construct a basis

$$|0\rangle \quad a_A^+|0\rangle \quad a_A^+\,a_B^+|0\rangle \qquad a_1^{+2}a_2^+|0\rangle = -a_2^+a_1^{+2}|0\rangle$$
$$a_1^+a_2^{+2}|0\rangle = -a_2^{+2}a_1^{\dagger}|0\rangle \qquad a_1^{+2}a_2^{+2}|0\rangle$$

which is actually 10 dimensional. Thus the related matrices β_μ are indeed reduced to 10×10 ones.

Formulae (6.17) present a realization of parafermionic creation and annihilation operators for $p = 2$ and the related para-Grassmanian variables in terms of 10×10 Duffin–Kemmer matrices. Inverting these formulae we obtain the realization of these matrices in terms of para-Grassmanian variables (refer to (6.10) and (6.14)).

6.2. Para-Grassmanian variables for arbitrary N and non-trivial central charges

Now let us consider the PPSA with arbitrary N and non-trivial central charges $Z_k < 2M$. In this case we need a set of N parafermionic creation and annihilation operators which will be defined as

$$a_A^{+j} = \sum_{\alpha=1}^{p} b_A^{+(\alpha)j} \qquad a_A^{j} = \sum_{\alpha=1}^{p} b_A^{(\alpha)j} \tag{6.18}$$

where $b_A^{+(\alpha)j}$ and $b_A^{(\alpha)j}$ are fermionic creation and annihilation operators (the index j labels the sort of fermion) satisfying the following relations:

$$\begin{aligned} &[b_A^{(\alpha)j}, b_B^{+(\alpha)j}]_+ = \delta_{AB}\delta_{ij} && [b_A^{(\alpha)j}, b_B^{(\alpha)j}]_+ = 0 \\ &[b_A^{(\alpha)j}, b_B^{+(\beta)j}] = [b_A^{(\alpha)j}, b_B^{(\beta)j}] = 0 && \alpha \neq \beta. \end{aligned} \tag{6.19}$$

In analogy with (6.4)–(6.9) we conclude that for any fixed j operators $b_A^{+(\alpha)j}$ can be realized as Grassmanian variables such that

$$b_A^{(\alpha)j} = \frac{\partial}{\partial\theta_A^{(\alpha)j}} \qquad b_A^{+(\alpha)j} = \theta_A^{(\alpha)j} \tag{6.20}$$

(variables with non-coinciding values of j should anticommute).

As in the case $N = 1$ we define parafermionic creation and annihilation operators and the related para-Grassmanian variables using the Green ansatz

$$a_A^{+j} = \theta_A^j = \sum_{\alpha_1}^{p} b_A^{+(\alpha)j} \qquad a_A^j = \frac{\partial}{\partial\theta_A^j} = \sum_{\alpha_1}^{p} b_A^{j(\alpha)}. \tag{6.21}$$

Such defined operators satisfy the following relations:

$$\begin{aligned} &[[(a_C^+)^{2k-1}, a_B^j], a_A^{2k-1}] = -2\delta_{AC}\delta_{km}a_B^j \\ &[[(a^+)^{2m}, a_B^j], a_A^{2k}] = -2\delta_{AC}\delta_{km}a_B^j. \end{aligned} \tag{6.22}$$

We again suppose that operators a_A^{+j}, a_A^j are defined in the Hilbert space with one cyclic vacuum, i.e.

$$a_A^j|0\rangle = 0 \qquad a_A^j a_B^{+\mathrm{i}}|0\rangle = p\delta_{AB}\delta_{ij}|0\rangle. \tag{6.23}$$

Consider now parasupercharges (3.9) which we represent as

$$\begin{aligned} &\hat{Q}_A^{2k-1} = \sqrt{2M - Z_k}\, a_A^{2k-1} && \hat{\overline{Q}}{}_A^{2k-1} = \sqrt{2M - Z_k}\, (a_A^+)^{2k-1} \\ &\hat{Q}_A^{2k} = \sqrt{2M + Z_k}\, a_A^{2k} && \hat{\overline{Q}}{}_A^{2k} = \sqrt{2M + Z_k}\, (a_A^+)^{2k} \end{aligned} \tag{6.24}$$

or, alternatively, with using para-Grassmanian variables

$$\begin{aligned} &\hat{Q}_A^{2k-1} = \sqrt{2M - Z_k}\, \frac{\partial}{\partial\theta_A^{2k-1}} && \hat{\overline{Q}}{}_A^{2k-1} = \sqrt{2M - Z_k}\, \theta_A^{2k-1} \\ &\hat{Q}_A^{2k} = \sqrt{2M + Z_k}\, \frac{\partial}{\partial\theta_A^{2k}} && \hat{\overline{Q}}{}_A^{2k} = \sqrt{2M + Z_k}\, \theta_A^{2k}. \end{aligned} \tag{6.25}$$

Relations (4.3)–(4.5), (3.9), (6.24), (6.25) present a realization of all basis elements of the extended PPSA with arbitrary N and non-trivial central charges $Z_m < 2M$ in terms of parafermionic creation and annihilation operators and para-Grassmanian variables.

Let us present a special realization of the extended PPSA with arbitrary N and trivial central charges which was called covariant [17, 19]. In this realization generators of the Poincaré group and parasupercharges have the following form:

$$P_\mu = p_\mu \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu} \tag{6.26}$$

$$\begin{aligned} Q_1^j &= S_{4N+1\ 4j-3} - \mathrm{i}S_{4N+1\ 4j-2} \\ Q_2^j &= -S_{4N+1\ 4j-1} + \mathrm{i}S_{4N+1\ 4j} \\ \bar{Q}_1^j &= (S_{4N+1\ 4j-3} - \mathrm{i}S_{4N+1\ 4j-2})(p_3 - p_0) + (S_{4N+1\ 4j-1} + \mathrm{i}S_{4N+1\ 4j})(p_1 - \mathrm{i}p_2) \\ \bar{Q}_2^j &= -(S_{4N+1\ 4j-1} + \mathrm{i}S_{4N+1\ 4j})(p_3 + p_0) + (S_{4N+1\ 4j-3} - \mathrm{i}S_{4N+1\ 4j-2})(p_1 + \mathrm{i}p_2). \end{aligned} \tag{6.27}$$

Here $S_{\mu\nu}$ are numerical matrices which commute with the orbital part of $J_{\mu\nu}$:

$$S_{ab} = \varepsilon_{abc}S_c^{(1)} \qquad S_{0a} = \mathrm{i}S_a^{(1)} \qquad a, b, c = 1, 2, 3 \tag{6.28}$$

and $S_a^{(1)}$ are matrices defined in (4.5).

It was shown in [19] that the realization (6.26)–(6.28) is equivalent to (4.3). As was done above, matrices $S_{\mu\nu}$ and parasupercharges Q_A^j, $\bar{Q}_A^j$ can be represented in terms of para-Grassmanian variables

$$\begin{aligned} S_{\mu\nu} &= \frac{1}{2}(\sigma_{\mu\nu})_{AB}\left[\theta_A^j, \frac{\partial}{\partial\theta_B^j}\right] \\ Q_A^j &= \frac{\partial}{\partial\theta_A^j} \qquad \bar{Q}_A^j = (\sigma_\mu)_{AB}\theta_B^j P^\mu \end{aligned} \tag{6.29}$$

where $\sigma_{0a} = \sigma_a$, $\sigma_{ab} = \frac{1}{2}(\sigma_a\sigma_b - \sigma_b\sigma_a)$, $a, b \neq 0$.

The representation (6.27) and (6.28) admits an important generalization to the case of complex Grassmanian variables. Namely, we can set

$$S_{\mu\nu} = \frac{1}{2}\sum_{j=1}^{N}\left((\sigma_{\mu\nu})_{AB}\left[\theta_A^j, \frac{\partial}{\partial\theta_B^j}\right] + (\sigma^\dagger{}_{\mu\nu})_{AB}\left[(\theta_A^j)^\dagger, \frac{\partial}{\partial(\theta_B^j)^\dagger}\right]\right) \tag{6.30}$$

$$Q_A^j = \frac{\partial}{\partial\theta_A^j} + (\sigma_\mu)_{AB}(\theta_B^j)^\dagger P^\mu \qquad \left(Q_A^j\right)^\dagger = \frac{\partial}{\partial(\theta_A^j)^\dagger} + \theta_B^j(\sigma_\mu)_{BA}P^\mu \tag{6.31}$$

where the conjugated variables generate zero double commutators with non-conjugated ones:

$$\left[\left[\frac{\partial}{\partial\theta_A^j}, \theta_B^k\right], (\theta_C^i)^\dagger\right] = \left[\left[\frac{\partial}{\partial\left(\theta_A^j\right)^\dagger}, \theta_B^k\right], (\theta_C^i)\right] = 0.$$

In an analogous way starting with representations with non-trivial charges (4.3)–(4.5), (3.9), (6.24), (6.25) we obtain the following realization for parasupercharges:

$$\begin{aligned} Q_A^i &= \sqrt{m}\frac{\partial}{\partial\theta_A^i} + \frac{1}{\sqrt{m}}(\sigma_\mu)_{AB}(\theta_B^i)^\dagger P^\mu + \frac{1}{\sqrt{m}}\widehat{Z}^{ij}\varepsilon_{AB}\theta_B^j \\ (Q_A^i)^\dagger &= \sqrt{m}\frac{\partial}{\partial(\theta_A^i)^\dagger} + \frac{1}{\sqrt{m}}\theta_B^i(\sigma_\mu)_{AB}P^\mu + \frac{1}{\sqrt{m}}\widehat{Z}^{ij}\varepsilon_{AB}(\theta_B^j)^\dagger \end{aligned} \tag{6.32}$$

where $\widehat{Z}^{ij}$ is the matrix whose elements are given in (3.8). The corresponding generators of the Poincaré group are still given by relations (6.26) and (6.30).

These are realizations (6.26)–(6.32) which will be used in section 8 to generate parasupersymmetric motion equations. We note that in contrast with the representation considered in sections 3–5, representations (6.26), (6.30)–(6.32) are reducible.

7. Invariant wave equations

7.1. Linear models

One possible application of the presented representations of the PPSA is the search for the related invariant equations, i.e. equations which are invariant with respect to both the Lorentz and parasupersymmetry transformations. To make such a search it is possible to use the well developed theory of relativistic wave equations (for a survey refer, for example, to [25]) and also knowledge of the spin content of parasupermultiplets [17, 19].

Here we present examples of such invariant wave equations.

Let us start with the simplest model compatible with PPSA (which is reduced in this case to PSA), i.e. with the famous Wess–Zumino (WZ) model [2]. It includes one spinor field φ and two spinor fields A_+ and A_-which satisfy the following equations:

$$\sigma_\mu p^\mu \varphi = 0 \qquad p_\nu p^\nu A_+ = 0 \qquad A_- = 0. \tag{7.1}$$

The system (7.1) can be rewritten as a single equation

$$L\psi = \left[S_{\mu\nu}S^{\mu\nu}(p_0 - 2S_a p_a) + (\Sigma + 1)\, p_\nu p^\nu + (\Sigma - 1)\,\kappa\right]\psi = 0 \tag{7.2}$$

where $\psi = \text{column}(\varphi_1, \varphi_2, A_+, A_-)$, $S_{\mu\nu}$ are matrices defined in (6.28) which in our case are reduced to the form

$$S_{oa} = \tfrac{1}{2}\mathrm{i}\varepsilon_{abc}S_{bc} = \mathrm{i}S_a \qquad S_a = \tfrac{1}{4}(\tfrac{1}{2}\mathrm{i}\varepsilon_{abc}\gamma_b\gamma_c + \gamma_0\gamma_a) \tag{7.3}$$

and

$$\Sigma = \tfrac{1}{2}(\gamma_1\gamma_2 - \gamma_0\gamma_3) \tag{7.4}$$

In addition, we choose a realization of the Dirac matrices γ_μ such that $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is diagonal and $(\gamma_5)_{11} = (\gamma_5)_{11} = 1$.

Equation (7.2) is invariant with respect to the algebra whose basis elements are given in (6.26)–(6.28). Indeed, in our case $Q_1 = \gamma_0 - \gamma_3$, $Q_1 = \gamma_1 + \mathrm{i}\gamma_2$, so it is easy verified that $[P_\mu, L]\psi = [J_{\mu\nu}, L]\psi = [Q_A, L]\psi = 0$. We note that Σ commutes with all operators (6.26).

Equations (7.1) correspond to the simplest case $p = 1$. To find invariant equations for $p = 2$ we define a tensor product T of the WZ multiplet $M = (\varphi_1, \varphi_2, \varphi_3 = A_+, \varphi_4 = 0)$ and a constant multiplet $C = (c_1, c_2, c_3, c_4 = 0)$, i.e.

$$T = (\varphi_1 c_1, \varphi_2 c_2, \ldots, \varphi_4 c_4). \tag{7.5}$$

Both M and C are carrier spaces of representations of the PPSA for $p = 1$. We denote the corresponding supercharges as

$$\begin{aligned}
&Q_1^{(\alpha)} = \gamma_0^{(\alpha)} - \gamma_3^{(\alpha)} \qquad Q_2 = \gamma_1^{(\alpha)} + \mathrm{i}\gamma_2^{(\alpha)} \qquad \alpha = 1, 2\\
&\bar{Q}_1^{(\alpha)} = \left(\gamma_0^{(\alpha)} - \gamma_3^{(\alpha)}\right)(p_3 - p_0) + \left(\gamma_1^{(\alpha)} + \mathrm{i}\gamma_2^{(\alpha)}\right)(p_1 - \mathrm{i}p_2)\\
&\bar{Q}_2^{(\alpha)} = -\left(\gamma_1^{(\alpha)} + \mathrm{i}\gamma_2^{(\alpha)}\right)(p_3 + p_0) + \left(\gamma_0^{(\alpha)} - \gamma_3^{(\alpha)}\right)(p_1 + \mathrm{i}p_2)
\end{aligned} \tag{7.6}$$

where $\alpha = 1$ and 2 relate to M and C, respectively, $\{\gamma_\mu^{(1)}\}$ and $\{\gamma_\mu^{(2)}\}$ are commuting sets of Dirac matrices satisfying (6.12). The corresponding generators of the Poincaré group are given by relations (6.26), where

$$S_{0a} = \tfrac{1}{2}\varepsilon_{abc}S_{bc} = \tfrac{1}{4}\sum_{\alpha=1}^{2}\left(\mathrm{i}\varepsilon_{abc}\gamma_b^{(\alpha)}\gamma_c^{(\alpha)} + \left[\gamma_0^{(\alpha)}, \gamma_a^{(\alpha)}\right]\right). \tag{7.7}$$

It can be easily shown that T is a carrier space of the representation of the PPSA with $p = 2$. The corresponding basis elements of the PPSA are given by (6.26), (7.7) and by the following relation:

$$Q_A = Q_A^{(1)} + Q_A^{(2)} \qquad \overline{Q}_A = \overline{Q}_A^{(1)} + \overline{Q}_A^{(2)} \tag{7.8}$$

where $Q_A^{(1)}, Q_A^{(2)}, \overline{Q}_A^{(1)}$ and $\overline{Q}_A^{(2)}$ are operators given in (7.6). This representation is reducible, refer to (6.14)–(6.16). To select the ten-dimensional irreducible representation we have to restrict ourselves to the symmetric part of the parasupermultiplet (7.5), i.e.

$$\begin{aligned} T^S = (T_1, T_2, \ldots, T_{10}) = (&\varphi_1 c_1, \varphi_2 c_2, \varphi_1 c_2 + \varphi_2 c_1, \varphi_1 c_3 + \varphi_3 c_1, \\ &\varphi_2 c_3 + \varphi_3 c_2, \varphi_3 c_3, \varphi_2 c_4 + \varphi_4 c_2, \varphi_4 c_1 + \varphi_1 c_4, \varphi_3 c_4 + \varphi_4 c_3, \varphi_4 c_4). \end{aligned} \tag{7.9}$$

A linear transformation $T_4 \rightarrow \widetilde{T}_4 = c_3 T_4 - c_1 T_6$, $T_5 \rightarrow \widetilde{T}_5 = c_3 T_5 - c_2 T_6$; $T_k \rightarrow \widetilde{T}_k = T_k$, $k \neq 4, 5$ reduces (7.9) to the form

$$\begin{aligned} \widetilde{T}^S = (\widetilde{T}_1, \widetilde{T}_2, \ldots, \widetilde{T}_{10}) = (&\varphi_1 c_1, \varphi_2 c_2, \varphi_1 c_2 + \varphi_2 c_1, \varphi_1 c_3^2, \varphi_2 c_3^2, \varphi_3 c_3, \\ &\varphi_2 c_4 + \varphi_4 c_2, \varphi_4 c_1 + \varphi_1 c_4, \varphi_3 c_4 + \varphi_4 c_3, \varphi_4 c_4). \end{aligned} \tag{7.10}$$

In addition, we extend (7.9) to the 11-dimensional multiplet

$$T^{SE} = (\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2, \ldots, \widetilde{T}_{10}) \tag{7.11}$$

where $\widetilde{T}_0 = c_4 \varphi_4$ and temporarily set $\widetilde{T}_0 = 0$. In other words, we reduce the 16×16-dimensional representation of the related Duffin–Kemmer matrices (6.14) to the direct sum of 10×10 and 1×1 (trivial) representations.

It follows from the definition of T^{SE} that $\widetilde{T}_1, \widetilde{T}_2, \ldots$ satisfy the equations

$$\begin{aligned} &\Sigma_\mu p^\mu \psi = 0 \\ &\sigma_\mu p^\mu \chi = 0 \\ &p_\mu p^\mu \widetilde{T}_6 = 0 \\ &\widetilde{T}_0 = \widetilde{T}_7 = \widetilde{T}_8 = \widetilde{T}_9 = \widetilde{T}_{10} = 0 \end{aligned} \tag{7.12}$$

where $\psi = \text{column}(\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3)$, $\chi = \text{column}(\widetilde{T}_4, \widetilde{T}_5)$, Σ_0 is the 4×4 unit matrix,

$$\Sigma_1 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}$$

$$\Sigma_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}$$

$$\Sigma_3 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}.$$

Formulae (7.12) present the simplest system of relativistic wave equations which admits $p = 2$ parasupersymmetry. It includes the Maxwell equation for vector $\widetilde{T}_a = H_a - \mathrm{i} E_a$ which satisfies the usual divergenceless condition $p_a \widetilde{T}_a = 0$, the Weil equation for the spinor χ and the D'Alembert equation for the complex scalar T_6. Invariance of this system with respect to the PPSA is obvious by construction. The related basis elements of the PPSA are given by (6.26)–(6.28), where

$$S_a^{(1)} = \frac{1}{2}\left(\mathrm{i}\varepsilon_{abc}\beta_b\beta_c + [\beta_0, \beta_a]\right) \qquad S_{5\mu} = \beta_\mu = \left(\begin{array}{c|c} 0 & \\ \hline & \widehat{\beta}_\mu \end{array}\right)$$

and $\widehat{\beta}_\mu$ are the 10×10 Duffin–Kemmer matrices whose non-zero elements are

$$\begin{aligned} 2\widehat{\beta}_0 = & -\mathrm{i}e_{1,7} + \mathrm{i}e_{1,10} + e_{2,7} + e_{2.10} + \mathrm{i}e_{3,8} + \mathrm{i}e_{3,9} - \sqrt{2}e_{4,10} + \mathrm{i}\sqrt{2}e_{5,7} \\ & -e_{6,8} - e_{6,9} + \mathrm{i}e_{7,1} + e_{7,2} - \mathrm{i}\sqrt{2}e_{7,5} - \mathrm{i}e_{8,3} - e_{8,6} - \mathrm{i}e_{9,3} - e_{9,6} \\ & -\mathrm{i}e_{10,1} + e_{10,2} - \sqrt{2}e_{10,4} \end{aligned}$$

$$\begin{aligned} 2\widehat{\beta}_1 = & \ \mathrm{i}e_{1,8} - \mathrm{i}e_{1,9} + e_{2,8} + e_{2,9} + \mathrm{i}e_{3,7} + \mathrm{i}e_{3,10} + \sqrt{2}e_{4,9} - \mathrm{i}\sqrt{2}e_{5,8} \\ & +e_{6,7} + e_{6,10} + \mathrm{i}e_{7,3} - e_{7,6} + \mathrm{i}e_{8,1} - e_{8,2} - \mathrm{i}\sqrt{2}e_{8,5} - \mathrm{i}e_{9,1} \\ & -e_{9,2} + \sqrt{2}e_{9,4} + \mathrm{i}e_{10,3} - e_{10,6} \end{aligned}$$

$$\begin{aligned} 2\widehat{\beta}_2 = & -e_{1,8} - \mathrm{i}e_{1,9} + \mathrm{i}e_{2,7} - \mathrm{i}e_{2,9} - e_{3,7} + e_{3,10} + \mathrm{i}\sqrt{2}e_{4,9} - \sqrt{2}e_{5,8} \\ & +\mathrm{i}e_{6,7} - \mathrm{i}e_{6,10} + e_{7,3} + \mathrm{i}e_{7,6} + e_{8,1} + \mathrm{i}e_{8,2} + \sqrt{2}e_{8,5} + e_{9,1} - \mathrm{i}e_{9,2} \\ & +\mathrm{i}\sqrt{2}e_{9,4} - e_{10,3} - \mathrm{i}e_{10,6} \end{aligned}$$

$$\begin{aligned} 2\widehat{\beta}_3 = & -\mathrm{i}e_{1,7} - \mathrm{i}e_{1,10} + e_{2,7} - e_{2.10} + \mathrm{i}e_{3,8} - \mathrm{i}e_{3,9} - \sqrt{2}e_{4,10} - \mathrm{i}\sqrt{2}e_{5,7} \\ & -e_{6,8} - e_{6,9} - \mathrm{i}e_{7,1} - e_{7,2} - \mathrm{i}\sqrt{2}e_{7,5} + \mathrm{i}e_{8,3} + e_{8,6} - \mathrm{i}e_{9,3} \\ & -e_{9,6} - \mathrm{i}e_{10,1} + e_{10,2} + \sqrt{2}e_{10,4}. \end{aligned}$$

Here $e_{k,l}$ denotes the unit element placed on row number k and line number l.

Wave equations for the $p = 2$ parasupermultiplet were first proposed in [14]. They differ from our formulation (7.10) by the absence of a zero divergence condition for $\widetilde{T}_a$ and so are not relativistic-invariant.

In an analogous manner, considering the tensor product of the multiplet (7.10) with a constant WZ supermultiplet and making a reduction to symmetric states, we obtain wave equations for the $p = 3$ parasupermultiplet, which includes the non-trivial vector-spinor. Ψ_B^c, $(B = 1, 2,\ c = 1, 2, 3)$, vector $\psi = \mathrm{column}(0, \psi_1, \psi_2, \psi_3)$, Weil spinor χ, scalar φ and, in addition, one vector, three spinor and five scalar fields which are identically zero. The related system of equations has the following form:

$$\left(\sigma_\mu p^\mu\right)_{AB} \Psi_B^c = 0 \qquad (p_0 + S_a p_a)_{bc} \Psi_B^c = 0 \qquad p_a \Psi_B^a = 0 \tag{7.13}$$

$$\Sigma_\mu p^\mu \psi = 0 \quad \sigma_\mu p^\mu \chi = 0 \qquad p_\mu p^\mu \varphi = 0 \tag{7.14}$$

were S_a are spin matrices for $s = 1$ whose elements are $(S_a)_{bc} = \mathrm{i}\varepsilon_{abc}$.

Equation (7.13) coincides with the equation for the massless field of spin $\frac{3}{2}$ proposed in [25]. The wavefunction Ψ_B^c satisfies the Weil equation with spinor index B and Maxwell equation with vector index c.

Continuing this procedure we can obtain equations for paramultiplets with $p = 4, 5$, etc. Moreover, in an analogous way it is possible to derive equations for massive parasupermultiplets. Thus, starting with the WZ equations

$$\sigma_\mu p^\mu \psi + m\overline{\psi} = 0 \qquad p_\mu p^\mu A_+ = m A_-^* \qquad A_- + m A_+^* = 0$$

we obtain equations for the $p = 2$ parasupermultiplet

$$\begin{aligned} &\Sigma_\mu p^\mu \psi = m\psi^* \\ &\sigma_\mu p^\mu \chi = -m\overline{\chi} \\ &p_\mu p^\mu \widetilde{T}_6 = m\widetilde{T}_9^* \\ &\widetilde{T}_0 = \widetilde{T}_{10} = 0 \qquad \widetilde{T}_7 + \widetilde{T}_8 = \lambda\widetilde{T}_6^* \qquad \lambda = \frac{c_1 + c_2}{c_3}. \end{aligned} \tag{7.15}$$

In contrast to the analogous system proposed in [14], equations (7.13) are relativistic invariant. The first of equations (7.13) is equivalent to a spin-1 equation in Dirac form [25].

7.2. Wess–Zumino–Weinberg model for arbitrary p, N and Z

Here we present a formal construction of nonlinear models which generalize both the WZ and Weinberg [26] approaches to the case of a parasuperfield with arbitrary p, N and Z.

We start with the case $N = 1$ and choose the realization of the PPSA in terms of para-Grassmanian variables (6.26), (6.30), (6.31) (the related index j takes the only value $j = 1$ and will be omitted temporarily). We define the corresponding representation space as a parasuperfield [27] $\Phi(x, \theta, (\theta)^\dagger)$ depending on the spatial variables x_μ and para-Grassmanian variables θ, $(\theta)^\dagger$. This space is reducible with respect to PPSA in as much as it is possible to impose on $\Phi(x, \theta, (\theta)^\dagger)$ one of the following invariant conditions:

$$\overline{D}_A \Phi(x, \theta, (\theta)^\dagger) = 0 \tag{7.16}$$

or

$$D_A \Phi(x, \theta, (\theta)^\dagger) = 0 \tag{7.17}$$

where

$$D_A = \frac{\partial}{\partial \theta_A} - (\sigma_\mu)_{AB}(\theta_B)^\dagger P^\mu \qquad \overline{D}_A = \frac{\partial}{\partial (\theta_A)^\dagger} - \theta_B(\sigma_\mu)_{AB} P^\mu \tag{7.18}$$

are covariant derivatives.

Operators $\varepsilon_{AB} D_A D_B = D_1 D_2 - D_2 D_1$ (and $\varepsilon_{AB}\overline{D}_A\overline{D}_B$) commute with any element of the PPSA and so relations (7.17) and (7.18) are invariant. To obtain a clear interpretation of (7.16) it is convenient to apply the transformation

$$\begin{aligned} &\Phi \to \Phi_+ = \exp(-G)\Phi \\ &D_A \to D'_A = \exp(G) D_A \exp(-G) = \frac{\partial}{\partial \theta_A} \end{aligned} \tag{7.19}$$

where $G = \frac{1}{2}(\sigma_\mu)_{AB}[(\theta_A)^\dagger, \theta_B]P^\mu$.

In accordance with (7.19) relation (7.16) reduces to $\frac{\partial}{\partial \theta_A}\Phi_+ = 0$, i.e. Φ_+ does not depend on θ.

We note that inverse transformation to (7.19) reduces $\overline{D}_A$ to the following form:

$$\overline{D'}_A = \exp(-G)]\overline{D}_A \exp(G) = \theta_A^\dagger. \tag{7.20}$$

Using these definitions we can present the invariant nonlinear equation as

$$\left([\overline{D}'_1, \overline{D}'_2]\right)^p \exp(-2G)\Phi_+^*(x, \theta) = m\Phi_+ + g\Phi_+^2(x, \theta) \tag{7.21}$$

where m is mass and g is an interaction constant.

For $p = 1$ and 2 we recover the supersymmetric Wess–Zumino model [2] and the model proposed in [14], respectively.

The Lagrangian which corresponds to equation (7.21) has the following form:

$$\mathcal{L} = (\Phi_+^* \exp(-2G)\Phi_+)_{\theta_1^p\theta_2^p(\theta_1^\dagger)^p(\theta_2^\dagger)^p} + \left(\tfrac{1}{2}m\Phi_+^2 + \tfrac{1}{3}g\Phi_+^3\right)_{\theta_1^p\theta_2^p} + \text{(h.c.)}.$$

The model proposed admits a straightforward generalization to the case of arbitrary N and non-trivial central charges. The related parasupercharges are defined in (6.32), while covariant derivatives are

$$\begin{aligned} D_A^i &= m\frac{\partial}{\partial\theta_A^i} - (\sigma_\mu)_{AB}(\theta_B^i)^\dagger P^\mu - \widehat{Z}^{ij}\varepsilon_{AB}\theta_A^j \\ \overline{D}_A^i &= m\frac{\partial}{\partial(\theta_A^i)^\dagger} - \theta_B^i(\sigma_\mu)_{AB}P^\mu - \varepsilon_{AB}\widehat{Z}^{ij}(\theta_A^j)^\dagger. \end{aligned} \tag{7.22}$$

Imposing the invariant conditions

$$\left(\varepsilon^{AB}D_A^jD_B^j\right)^p\Phi = 0$$

(no sum over j) on the related parasuperfield $\Phi = \Phi\left(\theta_B^i, \left(\theta_A^j\right)^\dagger, x\right)$ and applying the transformation (7.19) where

$$G = \frac{1}{2m}\left(\sum_{j=1}^{N}(\sigma_\mu)_{AB}\left[\left(\theta_A^j\right)^\dagger, \theta_B^j\right]P^\mu - \sum_{i,j=1}^{N}\widehat{Z}^{ij}\varepsilon_{AB}\left[\theta_A^i, \frac{\partial}{\partial(\theta_B^j)^\dagger}\right]\right) \tag{7.23}$$

we obtain the field Φ_+ which does not depend on θ_A^j. The corresponding invariant equation can again be written in the form (7.21) where G has a more complicated form (7.23).

Equation (7.23) can be treated as a parasupersymmetric analogue of the Weinberg equation [26] for a particle with arbitrary spin. Indeed, equation (7.23) includes only physical components present in the irreducible parasuperfield (no auxiliary fields are involved), and is a partial differential equation of order $2(2s + 1)$, where s is the maximal spin value for the parasupermultiplet. In addition, the operator $\exp(-2G)$ on the right-hand side of (7.22) is a direct generalization of the related Weinberg construction and transforms $J_{\mu\nu}$ to $J_{\mu\nu}^\dagger$.

8. Discussion

We present a description of IRs of the extended PPSA which includes ten generators of the Poincaré group, an arbitrary number N of parasupercharges, n central charges ($n \leqslant \{N/2\}$) and also an internal symmetry algebra which is $sp(n) \oplus u(N - 2n)$. Such rather complicated algebraic structure admits an explicit description in terms of generators of the little Wigner parasuperalgebra, which is equivalent to the direct sum of algebras $so(3)$ and $so(p, q)$ where $p = 4N - 2m_2 - 2m_3 + 1$, $q = 2m_2$; m_1, m_2 and m_3 are the numbers of central charges defined in (4.1).

In this way we complete investigations of IRs of the PPSA started in [17–19]. In particular, we obtain a parasupersymmetric analogue of known IRs of the PSA [22] which appear in our analysis as particular cases. Thus we present a new viewpoint on PSA which is only the simplest link in an infinite series of Poincaré parasuperalgebras.

A specific feature of our approach is that the basis elements of the related internal symmetry algebra are given explicitly both in terms of matrices belonging to (pseudo)orthogonal algebras and in terms of para-Grassmanian variables. In particular, such formulation can be useful for PSA and supersymmetric quantum field theory.

In addition, the extended PPSA admits such IRs which do not have analogues in the case of (extended) PSA. They are the representations which correspond to central charges larger than the doubled mass considered in subsections 4.3 and 5.3 above, and representations corresponding to negative eigenvalues of the Casimir operators $C_3 = \frac{P_0}{|P_0|}$ or $C_2 = P^\mu P_\mu$, described in [17–19].

We note that realizations in terms of para-Grassmanian variables which are presented in section 6 for the case $0 < Z_m < 2M$ only, admit straightforward extensions to the limiting cases $Z_m \equiv 0$ and $Z_m = 2M$ (in the last case operators a_A^{2k-1} and a_A^{+2k-1} are zero in as much as, in accordance with (4.16), it corresponds to $\hat{Q}_A^{2k-1} = 0$). A natural question arises as to whether it is possible to formulate the analogous results for the case $Z_m > 2M$.

The possibility of describing the considered classes of IRs of the extended PPSA using parafermionic creation and annihilation operators or para-Grassmanian variables is a simple consequence of the well known fact that orthogonal algebras $so(n)$ are isomorphic to algebras (7.5) [28]. In the case $Z_m > 2M$ representations of the PPSA are induced by pseudo-orthogonal algebras (4.14) whose Hermitian representations are infinite dimensional. Such algebras cannot be realized via a finite number of fermionic creation and annihilation operators; however, it is possible to construct the corresponding representations in terms of (para)bosonic creation and annihilation operators [7].

We also present linear and nonlinear models which are invariant with respect to the PPSA. The generalized parasupersymmetric WZ–Weinberg model is a rather straightforward and simple extension of the famous WZ model [2]. However, it can lead to undesired complications which are typical to the Weinberg approach and are connected with the presence of higher derivatives. Thus it is interesting to search for other models which admit this specific combination of symmetries, i.e. relativistic invariance and parasupersymmetry.

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